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Fermionic approach to the evaluation of integrals of rational symmetric functions [1]

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Abstract

We use the fermionic construction of two-matrix model partition functions to evaluate integrals over rational symmetric functions. This approach is complementary to the one used in the paper "Integrals of Rational Symmetric Functions, Two-Matrix Models and Biorthogonal Polynomials" [1], where these integrals were evaluated by a direct method. Using Wick's theorem, we obtain the same determinantal expressions in terms of biorthogonal polynomials as in [1].

1 Introduction

In this work, we consider the following integral

$$\mathbf{I}_N^{(2)}(\xi,\zeta,\eta,\mu) \;\; := \;\; \frac{1}{\mathbf{Z}_N^{(2)}}\int\int d\mu(x_1,y_1)\dots\int\int d\mu(x_N,y_N)\Delta_N(x)\Delta_N(y)$$

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$$\times \prod_{a=1}^{N} \frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha - x_a) \prod_{\beta=1}^{L_2}(\zeta_\alpha - y_a)}{\prod_{j=1}^{M_1}(\eta_j - x_a) \prod_{k=1}^{M_2}(\mu_k - y_a)}, \tag{1.1}$$

where

$$\Delta_N(x) := \prod_{i>j}^{N}(x_i - x_j), \quad \Delta_N(y) := \prod_{i>j}^{N}(y_i - y_j) \tag{1.2}$$

are Vandermonde determinants, and

$$\mathbf{Z}_N^{(2)} := \int\int d\mu(x_1, y_1) \dots \int\int d\mu(x_N, y_N)\Delta_N(x)\Delta_N(y), \tag{1.3}$$

is a normalization constant, which in the context of two-matrix models is interpreted as the partition function. Here $d\mu(x, y)$ is a measure (in general, complex), supported on a finite set of products of curves in the complex x and y planes.

It is known (see for instance [5]) that this integral can be presented in the form of a determinant of an $N \times N$ matrix; namely,

$$\mathbf{I}_N^{(2)}(\xi, \zeta, \eta, \mu) = \frac{N!}{\mathbf{Z}_N^{(2)}} \det \left(\int\int \frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha - x) \prod_{\beta=1}^{L_2}(\zeta_\beta - y)}{\prod_{j=1}^{M_1}(\eta_j - x) \prod_{k=1}^{M_2}(\mu_k - y)} d\mu(x, y)(x, y)x^j y^k \right)_{0 \le j,k \le N-1}. \tag{1.4}$$

However for some purposes, it is more useful to express it as the determinant of a matrix whose size does not depend on N.

The problem of evaluation of integrals of such symmetric functions of N variables is of importance in the context of matrix models [4], where N is the number of eigenvalues. Expressing integrals that determine correlation functions and expectation values as determinants of matrices whose size is independent of N is of importance e.g., in the study of $N \to \infty$ limits. For one-matrix models such integrals were studied in [3], [6], [7], [8] and [9]. In the case of complex, normal or two-matrix models [10] the problem was considered in [8], [2], [1].

Remark 1.1. In order to compare with the results in [8] and [2] on complex integrals the variables $\{x_a, y_a\}_{a=1,\dots N}$ must be replaced by complex conjugate pairs $\{z_a, \bar{z}_a\}_{a=1,\dots N}$, and the integration domains taken as N copies of the complex plane.

The present paper is complementary to ref. [1], where such integrals were evaluated by a "direct" method based on partial fraction expansions and the Cauchy-Binet identity.

Ref. [1] and the present work are intended to provide concise presentations and new derivations of previously known as well as new results, using two distinct methods.

The approach used here is based on constructing a fermionic representation of integrals of such rational symmetric functions, introduced in [19], using two–component fermions. This representation is different from the ones used previously in the context of matrix models in [16], [17] and [18], which were based on one-component fermions.

Remark 1.2. Although there is a close connection between matrix integrals, orthogonal polynomials and the spectral transform approach to the theory of integrable systems, in the present work we do not develop this latter aspect, which concerns the deformation theory of the measures involved. Our starting point however is similar to the previous paper [19], where the deformation theory is addressed, and the fermionic approach to integrable systems developed in [14] is utilized. The relation of these results to the objects appearing in the spectral transform approach is, however, briefly explained in the Appendices, where the biorthogonal polynomials and their Hilbert transforms are interpreted as Baker functions and adjoint Baker functions.

The language of free two-component fermions used here is borrowed from [14]. The integrals that we are interested in are expressed as vacuum state expectation values of operator products formed from free fermion generators. Besides successive application of Wick's theorem, the main computational device used consists of applying canonical ("dressing") transformations to pass from free fermions fields, whose Laurent coefficients are the free Fermi creation and annihilation operators, to "dressed' ones, where the monomials are replaced by biorthogonal polynomials. Wick's theorem then serves to express the same operator product vacuum state expectation value (VEV) both as a multiple rational integral and as the determinant of a matrix formed from elementary factors involving orthogonal polynomials and their Hilbert transforms.

Let $d\mu(x, y)$ be a measure (in general, complex), supported on a finite set of products of curves in the complex x and y planes, for which the semi-infinite matrix of bimoments is finite:

$$B_{jk} := \int \int d\mu(x, y) x^j y^k < \infty, \quad 0, \quad \forall j, k \in \mathbb{N}. \tag{1.5}$$

The integrals are understood to be evaluated on a specified linear combination of products of the support curves. Assuming that, for all $N \geq 1$, the $N \times N$ submatrix $(B_{jk})_{0 \leq j,k, \leq N-1}$ is nonsingular, the Gram-Schmidt process may be used to construct an infinite sequence of pairs of biorthogonal polynomials $\{P_j(x), S_j(y)\}_{j=0 \ldots \infty}$, unique up to signs, satisfying

$$\int \int d\mu(x, y) P_j(x) S_k(y) = \delta_{jk}, \tag{1.6}$$

and normalized to have leading coefficients that are equal:

$$P_j(x) = \frac{x^j}{\sqrt{h_j}} + O(x^{j-1}), \qquad S_j(x) = \frac{y^j}{\sqrt{h_j}} + O(y^{j-1}). \tag{1.7}$$

We will also assume that the Hilbert transforms of these biorthogonal polynomials,

$$\tilde{P}_n(\mu) := \int\int d\mu(x,y)\frac{P_m(x)}{\mu - y}, \quad \tilde{S}_n(\eta) := \int\int d\mu(x,y)\frac{S_m(x)}{\eta - x}, \tag{1.8}$$

exist for all $n \in \mathbb{N}$.

The following expression for $\mathbf{Z}_N^{(2)}$ in terms of the leading term normalization factors h_n is then easily shown to hold (see, e.g., [4]).

$$\mathbf{Z}_N^{(2)} = N!\det\left(B_{jk}\right)|_{j,k=1,\dots,N} = N!\prod_{n=0}^{N-1} h_n \tag{1.9}$$

Defining

$$N_1 := N + L_1 - M_1, \qquad N_2 := N + L_2 - M_2, \tag{1.10}$$

we consider different cases. In each case the answer is written in form of the determinant of a matrix G which is different for different cases. Results are given

(1) $N_2 \geq N_1 \geq 0$: by formulae (5.9)-(5.15). G is a $(L_2 + M_1) \times (L_2 + M_1)$ matrix

(2) $N_1 \leq 0 \leq N_2$: by formulae (5.34)-(5.35). G is a $(L_2 + M_1) \times (L_2 + M_1)$ matrix

(3) $N_1 \leq N_2 \leq 0$: by formulae (5.44)-(5.45). G is a $(M_1 + M_2 - N) \times (M_1 + M_2 - N)$ matrix

The cases $N_1 \geq N_2 \geq 0$, $N_2 \leq 0 \leq N_1$ and $N_2 \leq N_1 \leq 0$ are related to previous ones by interchanging quantities $L_1 \leftrightarrow L_2$, $M_1 \leftrightarrow M_2$, $N_1 \leftrightarrow N_2$, $\xi \leftrightarrow \zeta$, $\mu \leftrightarrow \eta$, $P_n \leftrightarrow S_n$ and $\tilde{P}_n \leftrightarrow \tilde{S}_n$ in the final formulae. (See respectively (5.9)-(5.15) vs. (5.22)-(5.26), (5.34)-(5.35) vs. (5.36)-(5.37) and (5.44)-(5.45) vs. (5.46)-(5.47).)

For instance, for the case $N_1 \geq N_2 \geq 0$ the answer is

$$\begin{aligned}\mathbf{I}_N^{(2)}(\xi,\zeta,\eta,\mu) \;&=\; (-1)^{\frac{1}{2}(M_1+M_2)(M_1+M_2-1)}(-1)^{L_1M_2}\prod_{n=0}^{N-1} h_n^{-1} \prod_{n=N}^{N+L_2-M_2-1}\sqrt{h_n}\prod_{n=N}^{N+L_1-M_1-1}\sqrt{h_n} \\ &\times \frac{\prod_{\alpha=1}^{L_1}\prod_{j=1}^{M_1}(\xi_\alpha - \eta_j)\prod_{\beta=1}^{L_2}\prod_{k=1}^{M_2}(\zeta_\beta - \mu_k)}{\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)}\,\det G,\end{aligned} \tag{1.11}$$

where G is a $(L_1{+}M_2){\times}(L_1{+}M_2)$ matrix whose entries are expressed in terms of the values of $P_n, S_n, \tilde{P}_n$ and $\tilde{S}_n$ for $n = 0, \dots, \max(N_1, N_2)$ evaluated at the points $\{\xi_\alpha, \zeta_\beta, \eta_j, \mu_k\}$.

Remark 1.3. It is assumed that the support curves for the integrals involved do not pass through the points $\{\eta_j, \mu_k\}_{j=1,\dots M_1, k=1,\dots M_2}$. The definition of the integrals may be extended throughout the complex plane of these parameters by analytic continuation, but might then take on multiple values. However, if the measures $d\mu(x,y)$ vanish with sufficient rapidity as these points approach the support curves, the integrals considered in the present paper become single valued. If the replacements $\{x_a, y_a\}_{a=1,\dots N} \rightarrow \{z_a, \bar{z}\}_{a=1,\dots N}$, are made, in order to compare with the results of [8], [2], it must be assumed that the corresponding measures $d\mu(z, \bar{z}$ in the complex plane vanish sufficiently rapidly at the points $\{z = \eta_j, \bar{z} = \mu_k\}$ for the integrals to converge.

In the following sections, we present the integral (1.1) as a vacuum state expectation value of a certain fermionic expression, using two-component fermions (see (3.9) below). To evaluate this expectation value we use Wick's theorem, which expresses the vacuum state expectation value of products of linear combinations of fermions as a determinant of the matrix formed by evaluating the pair-wise VEV's. Since the expression (3.9) is not as yet of the form of a vacuum expectation value of a product of fermions, we use a set of tricks (namely, canonical transformations of fermions and re-writing of charged vacuum vectors in a suitable form) which serve also to reduce the final number of fermions inside the vacuum expectation value, and therefore to reduce as much as possible the size of the matrix G.

The structure of the paper is the following. First we recall some facts about free fermions, including Wick's theorem. Then we equal $\mathbf{I}_N^{(2)}(\xi, \zeta, \eta, \mu)$ to a certain vacuum expectation value, see (3.12). After certain preliminaries, via (4.25), we introduce dressed fermions $b_\alpha, \bar{b}_\alpha$ to adapt formula (3.12) for a usage of Wick's theorem in its determinantal form. Then we consider each of the numbered cases separately. For the case (1) (which is the most involved case) we introduce fermionic operators $a_\alpha, \bar{a}_\alpha$, obtained from $b_\alpha, \bar{b}_\alpha$ via a sort of conjugation (see (5.4)-(5.7)), and whose vacuum expectation evaluated via Wick's theorem yields the final result given by formulae (5.9)-(5.15) and are illustrated by six examples. For the case (2), instead of $a_\alpha, \bar{a}_\alpha$, we similarly use operators $b_\alpha, \bar{b}_\alpha$. At last, for the case (3), instead of $a_\alpha, \bar{a}_\alpha$, we exploit operators $c_\alpha, \bar{c}_\alpha$ introduced in (5.39)-(5.42).

Remark 1.4. One can obtain the corresponding results for N-fold integrals arising in one-matrix models by specifying that the measure be proportional to Dirac delta function, say $\delta(x-y)$. As this specialization will not be consider detailed here we shall write $\mathbf{I}_N$ instead of $\mathbf{I}_N^{(2)}$.

2 Summary of free fermions

The following is a summary regarding the one and two-component free fermion algebra based on the introductory section of [19]. The reader may refer to [19], or [15], [14] for further details.

2.1 One component fermions

In the following, $\mathcal{A}$ denotes the complex Clifford algebra over $\mathbb{C}$ generated by *charged free fermions* $\{f_i, \bar{f}_i\}_{i\in\mathbf{Z}}$, satisfying the anticommutation relations

$$[f_i, f_j]_+ = [\bar{f}_i, \bar{f}_j]_+ = 0, \quad [f_i, \bar{f}_j]_+ = \delta_{ij}. \tag{2.1}$$

where $[,]_+$ denotes the anticommutator.

Elements of the linear part

$$W := (\oplus_{m\in\mathbf{Z}}\mathbb{C}f_m) \oplus \left(\oplus_{m\in\mathbf{Z}}\mathbb{C}\bar{f}_m\right) \tag{2.2}$$

will be referred to as a *free fermions.* The fermionic free fields

$$f(x) := \sum_{k\in\mathbf{Z}} f_k x^k, \quad \bar{f}(y) := \sum_{k\in\mathbf{Z}} \bar{f}_k y^{-k-1}, \tag{2.3}$$

may be viewed as generating functions for the $f_j, \bar{f}_j$'s.

This Clifford algebra has a standard Fock space representation F and dual space $\bar{F}$ (see [19]) which contain unique vacuum states $|0\rangle$ and $\langle 0|$ respectively satisfying the properties

$$\begin{aligned} &f_m|0\rangle = 0 \qquad (m<0), \qquad &\bar{f}_m|0\rangle = 0 \qquad (m \geq 0),\\ &\langle 0|f_m = 0 \qquad (m \geq 0), \qquad &\langle 0|\bar{f}_m = 0 \qquad (m<0). \end{aligned} \tag{2.4}$$

The *Fock spaces* F and $\bar{F}$ are mutually dual, with the hermitian pairing defined via the linear form $\langle 0||0\rangle$ on $\mathcal{A}$ called the *vacuum expectation value.* This satisfies

$$\langle 0|1|0\rangle = 1; \quad \langle 0|f_m\bar{f}_m|0\rangle = 1, \quad m<0; \quad \langle 0|\bar{f}_m f_m|0\rangle = 1, \quad m \geq 0, \tag{2.5}$$

$$\langle 0|f_n|0\rangle = \langle 0|\bar{f}_n|0\rangle = \langle 0|f_m f_n|0\rangle = \langle 0|\bar{f}_m\bar{f}_n|0\rangle = 0; \quad \langle 0|f_m\bar{f}_n|0\rangle = 0, \quad m \neq n,. \tag{2.6}$$

Wick's theorem implies that for any finite set of elements $\{w_k \in W\}$, we have

$$\langle 0|w_1 \cdots w_{2n+1}|0\rangle = 0,$$

$$\langle 0|w_1 \cdots w_{2n}|0\rangle = \sum_{\sigma \in S_{2n}} sgn\sigma \langle 0|w_{\sigma(1)} w_{\sigma(2)}|0\rangle \cdots \langle 0|w_{\sigma(2n-1)} w_{\sigma(2n)}|0\rangle. \tag{2.7}$$

Here σ runs over permutations for which $\sigma(1) < \sigma(2), \ldots, \sigma(2n-1) < \sigma(2n)$ and $\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1)$.

If $\{w_i\}_{i=1,\ldots,N}$, are linear combinations of the f_j's only, $j \in \mathbb{Z}$, and $\{\bar{w}_i\}_{i=1,\ldots,N}$ linear combinations of the $\bar{f}_j$'s, $j \in \mathbb{Z}$, then(2.7) implies

$$\langle 0|w_1 \cdots w_N \bar{w}_N \cdots \bar{w}_1|0\rangle = \det\left(\langle 0|w_i \bar{w}_j|0\rangle\right)|_{i,j=1,\ldots,N} \tag{2.8}$$

Following [14], [15], for all $N \in \mathbb{Z}$, we also introduce the states

$$\langle N| := \langle 0|C_N \tag{2.9}$$

where

$$C_N := \bar{f}_0 \cdots \bar{f}_{N-1} \quad \text{if } N > 0 \tag{2.10}$$

$$C_N := f_{-1} \cdots f_N \quad \text{if } N < 0 \tag{2.11}$$

$$C_N := 1 \quad \text{if } N = 0 \tag{2.12}$$

and

$$|N\rangle := \bar{C}_N|0\rangle \tag{2.13}$$

where

$$\bar{C}_N := f_{N-1} \cdots f_0 \quad \text{if } N > 0 \tag{2.14}$$

$$\bar{C}_N := \bar{f}_N \cdots \bar{f}_{-1} \quad \text{if } N < 0 \tag{2.15}$$

$$\bar{C}_N := 1 \quad \text{if } N = 0 \tag{2.16}$$

The states (2.9) and (2.13) are referred to as the left and right charged vacuum vectors, respectively, with charge N.

From the relations

$$\langle 0|\bar{f}_{N-k} f(x_i)|0\rangle = x_i^{N-k}, \quad \langle 0|f_{-N+k-1}\bar{f}(y_i)|0\rangle = y_i^{N-k}, \quad k = 1, 2, \ldots N, \tag{2.17}$$

and (2.8), it follows that

$$\langle N|f(x_n) \cdots f(x_1)|0\rangle = \delta_{n,N}\Delta_N(x), \quad N \in \mathbb{Z}, \tag{2.18}$$

$$\langle -N|\bar{f}(y_n) \cdots \bar{f}(y_1)|0\rangle = \delta_{n,N}\Delta_N(y), \quad N \in \mathbb{Z}. \tag{2.19}$$

For free fermion generators with $|x| \neq |y|$,

$$\langle 0|f(x)\bar{f}(y)|0\rangle = \frac{1}{x-y} \tag{2.20}$$

Note that the expression on the right hand side is actually defined, by (2.5), as the infinite series $\sum_{n=0}^{\infty} y^n x^{-n-1}$ which converges only inside $|x| < |y|$. However one can consider expression (2.20) for the whole region of x and y (when $|x| \neq |y|$) in the sense of analytical continuation.

From Wick's theorem it follows that

$$\langle n-m|f(x_n)\cdots f(x_1)\bar{f}(y_m)\cdots \bar{f}(y_1)|0\rangle = \frac{\Delta_n(x)\Delta_m(y)}{\prod_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}}(x_i - y_j)} \tag{2.21}$$

2.2 Two-component fermions

The two-component fermion formalism is obtained by relabelling the above as follows.

$$f_n^{(\alpha)} := f_{2n+\alpha-1}\ , \qquad \bar{f}_n^{(\alpha)} := \bar{f}_{2n+\alpha-1}\ , \tag{2.22}$$

$$f^{(\alpha)}(z) := \sum_{k=-\infty}^{+\infty} z^k f_k^{(\alpha)}\ , \quad \bar{f}^{(\alpha)}(z) := \sum_{k=-\infty}^{+\infty} z^{-k-1} \bar{f}_k^{(\alpha)}\ , \tag{2.23}$$

where $\alpha = 1, 2$. Then (2.1) is equivalent to

$$[f_n^{(\alpha)}, f_m^{(\beta)}]_+ = [\bar{f}_n^{(\alpha)}, \bar{f}_m^{(\beta)}]_+ = 0, \qquad [f_n^{(\alpha)}, \bar{f}_m^{(\beta)}]_+ = \delta_{\alpha,\beta}\delta_{nm}. \tag{2.24}$$

We denote the right and left vacuum vectors respectively as

$$|0,0\rangle := |0\rangle, \quad \langle 0,0| := \langle 0|. \tag{2.25}$$

Relations (2.4) then become, for $\alpha = 1, 2$,

$$f_m^{(\alpha)}|0,0\rangle = 0 \qquad (m < 0), \qquad \bar{f}_m^{(\alpha)}|0,0\rangle = 0 \qquad (m \geq 0), \tag{2.26}$$

$$\langle 0,0|f_m^{(\alpha)} = 0 \qquad (m \geq 0), \qquad \langle 0,0|\bar{f}_m^{(\alpha)} = 0 \qquad (m < 0). \tag{2.27}$$

For $n, m \in \mathbb{Z},\ i, j = 1, 2$ it follows from (2.5)-(2.6) that

$$\langle 0,0|f_n^{(i)} f_m^{(j)}|0,0\rangle = \langle 0,0|\bar{f}_n^{(i)} \bar{f}_m^{(j)}|0,0\rangle = 0, \tag{2.28}$$

$$\langle 0,0|f_n^{(i)} \bar{f}_m^{(j)}|0,0\rangle = \delta_{ij}\delta_{nm},\ n < 0 \tag{2.29}$$

Following [14], [15], we also introduce the states

$$\langle n^{(1)}, n^{(2)}| := \langle 0,0|C_{n^{(1)}}C_{n^{(2)}}, \tag{2.30}$$

where

$$C_{n^{(\alpha)}} := \bar{f}_0^{(\alpha)} \cdots \bar{f}_{n^{(\alpha)}-1}^{(\alpha)} \quad \text{if } n^{(\alpha)} > 0 \tag{2.31}$$

$$C_{n^{(\alpha)}} := f_{-1}^{(\alpha)} \cdots f_{n^{(\alpha)}}^{(\alpha)} \quad \text{if } n^{(\alpha)} < 0 \tag{2.32}$$

$$C_{n^{(\alpha)}} := 1 \quad \text{if } n^{(\alpha)} = 0 \tag{2.33}$$

and

$$|n^{(1)}, n^{(2)}\rangle := \bar{C}_{n^{(2)}}\bar{C}_{n^{(1)}}|0,0\rangle \tag{2.34}$$

where

$$\bar{C}_{n^{(\alpha)}} := f_{n^{(\alpha)}-1}^{(\alpha)} \cdots f_0^{(\alpha)} \quad \text{if } n^{(\alpha)} > 0 \tag{2.35}$$

$$\bar{C}_{n^{(\alpha)}} := \bar{f}_{n^{(\alpha)}}^{(\alpha)} \cdots \bar{f}_{-1}^{(\alpha)} \quad \text{if } n^{(\alpha)} < 0 \tag{2.36}$$

$$\bar{C}_{n^{(\alpha)}} := 1 \quad \text{if } n^{(\alpha)} = 0 \tag{2.37}$$

The states (2.30) and (2.34) will be referred to, respectively, as left and right charged vacuum vectors with charges $(n^{(1)}, n^{(2)})$.

It follows that

$$f_m^{(1)}|n,*\rangle = 0 \qquad (m < n), \qquad \bar{f}_m^{(1)}|n,*\rangle = 0 \qquad (m \geq n), \tag{2.38}$$

$$\langle n,*|f_m^{(1)} = 0 \qquad (m \geq n), \qquad \langle n,*|\bar{f}_m^{(1)} = 0 \qquad (m < n), \tag{2.39}$$

and

$$f_m^{(2)}|*,n\rangle = 0 \qquad (m < n), \qquad \bar{f}_m^{(2)}|*,n\rangle = 0 \qquad (m \geq n), \tag{2.40}$$

$$\langle *,n|f_m^{(2)} = 0 \qquad (m \geq n), \qquad \langle *,n|\bar{f}_m^{(2)} = 0 \qquad (m < n). \tag{2.41}$$

Remark 2.1. Note that if we shift the charges of the vacuum vectors

$$\langle *,*| \to \langle *,*+n|, \quad |*,*\rangle \to |*,*+n\rangle \tag{2.42}$$

and, at the same time, re-label

$$f_i^{(2)} \to f_{i+n}^{(2)}, \quad \bar{f}_i^{(2)} \to \bar{f}_{i+n}^{(2)}, \quad i,n \in \mathbb{Z}, \tag{2.43}$$

then the vacuum expectation values remain invariant.

Remark 2.2. Wick's theorem will be used in the form (2.8) in the following. In the two component notation, there are two possible ways to do this; either

(1) Use formula (2.8), remembering that 2-component fermions consist just of the usual even and odd ones (see (2.22)), or

(2) Use formula (2.8) separately for each component. To calculate the vacuum state expectation value of an operator O, we first present it in the form

$$O = \sum_i O_i^{(1)} O_i^{(2)}. \tag{2.44}$$

Then

$$\langle 0,0|O|0,0\rangle = \sum_i \langle 0,0|O_i^{(1)} O_i^{(2)}|0,0\rangle = \sum_i \langle 0,0|O_i^{(1)}|0,0\rangle\langle 0,0|O_i^{(2)}|0,0\rangle \tag{2.45}$$

where Wick's theorem in the form (2.8) may be applied to each factor $\langle 0,0|O_i^{(\alpha)}|0,0\rangle$

The version (2) is used in the Section 3, and the version (1) is used in the Section 5.

Now define

$$\begin{aligned} F^{(j)}(x^{(j)}) &:= f^{(j)}(x^{(j)}_{n_j}) \cdots f^{(j)}(x^{(j)}_1), \\ \bar{F}^{(j)}(y^{(j)}) &:= \bar{f}^{(j)}(y^{(j)}_{m_j}) \cdots \bar{f}^{(j)}(y^{(j)}_1), \quad j = 1,2 \end{aligned} \tag{2.46}$$

Combining (2.21) and (2.45) we obtain

$$\begin{aligned} &\langle n_1 - m_1, n_2 - m_2 | F^{(2)}(x^{(2)}) F^{(1)}(x^{(1)}) \bar{F}^{(1)}(y^{(1)}) \bar{F}^{(2)}(y^{(2)}) |0,0\rangle \\ &= (-1)^{m_2(m_1+n_1)} \frac{\Delta_{n_1}(x^{(1)})\Delta_{m_1}(y^{(1)})\Delta_{n_2}(x^{(2)})\Delta_{m_2}(y^{(2)})}{\prod_{\substack{i=1,\dots,n_1 \\ j=1,\dots,m_1}} (x_i^{(1)} - y_j^{(1)}) \prod_{\substack{i=1,\dots,n_2 \\ j=1,\dots,m_2}} (x_i^{(2)} - y_j^{(2)})}. \end{aligned} \tag{2.47}$$

expectation value

3 The integral of rational symmetric functions as a certain expectation value

We shall use the following notations

$$\eta = (\eta_1, \dots, \eta_{M_1}), \quad \xi = (\xi_1, \dots, \xi_{L_1}), \quad \mu = (\mu_1, \dots, \mu_{M_2}), \quad \zeta = (\zeta_1, \dots, \zeta_{L_2}) \tag{3.1}$$

$$N_j = N + L_j - M_j, \quad j = 1,2, \tag{3.2}$$

$$\int d\mu(\mathbf{x}, \mathbf{y})(\cdot) = \int\int d\mu(x_1, y_1) \dots \int\int d\mu(x_N, y_N)(\cdot) \tag{3.3}$$

Let

$$F^{(1)}(\xi) := f^{(1)}(\xi_{L_1}) \cdots f^{(1)}(\xi_1), \tag{3.4}$$

$$F^{(2)}(\mu) := f^{(2)}(\mu_{M_2}) \cdots f^{(2)}(\mu_1), \tag{3.5}$$

$$\bar{F}^{(1)}(\eta) := \bar{f}^{(1)}(\eta_{M_1}) \cdots \bar{f}^{(1)}(\eta_1), \tag{3.6}$$

$$\bar{F}^{(2)}(\zeta) := \bar{f}^{(2)}(\zeta_{L_2}) \cdots \bar{f}^{(2)}(\zeta_1), \tag{3.7}$$

and

$$g := e^{A}, \quad A := \int\int f^{(1)}(x)\bar{f}^{(2)}(y) d\mu(x,y), \tag{3.8}$$

and consider the expression

$$J_N(\xi,\zeta,\eta,\mu) := R_N \langle N_1, -N_2 | F^{(2)}(\mu) F^{(1)}(\xi)\ g\ \bar{F}^{(1)}(\eta)\bar{F}^{(2)}(\zeta)|0,0\rangle \tag{3.9}$$

where

$$R_N = R_N(\xi,\zeta,\eta,\mu) := \frac{s(L_1,L_2,M_1,M_2)}{\prod_{n=0}^{N-1} h_n} \frac{\prod_{\alpha=1}^{L_1}\prod_{j=1}^{M_1}(\xi_\alpha - \eta_j)\prod_{\beta=1}^{L_2}\prod_{k=1}^{M_2}(\zeta_\beta - \mu_k)}{\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)}, \tag{3.10}$$

and $s(L_1,L_2,M_1,M_2)$ is the sign factor

$$s(L_1,L_2,M_1,M_2)) = (-1)^{\frac{1}{2}N(N+1)+L_2(L_1+M_1)+N(L_1+M_1)+M_2L_2} \tag{3.11}$$

Remark 3.1. Because of the form (3.8), g commutes with both $F^{(1)}(\xi)$ and $\bar{F}^{(2)}(\mu)$. Moreover he conditions given in Remark (1.3) also imply that g commutes with $\bar{F}^{(1)}(\eta)$ and $F^{(2)}(\zeta)$.

The main result of this subsection is the equality

$$\mathbf{I}_N(\xi,\zeta,\eta,\mu) = J_N(\xi,\zeta,\eta,\mu) \tag{3.12}$$

Proof. Inserting

$$g = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

into (3.9), we note that only the N-th power contributes, giving

$$J_N(\xi,\zeta,\eta,\mu) = \frac{R_N}{N!}\langle N_1, -N_2|F^{(2)}(\mu)F^{(1)}(\xi)A^N\bar{F}^{(1)}(\eta)\bar{F}^{(2)}(\zeta)|0,0\rangle \tag{3.13}$$

Collecting the fermion terms of the same types (which gives rise to a sign factor), the right hand side of (3.13) may be expressed as

$$(-1)^{\frac{1}{2}N(N-1)+NM_1}\frac{R_N}{N!}\int d\mu(\mathbf{x},\mathbf{y})\langle N_1,-N_2|F^{(2)}(\mu)F^{(1)}(\xi,x)\bar{F}^{(1)}(\eta)\bar{F}^{(2)}(y,\zeta)|0,0\rangle$$

where $(\xi, x) = (\xi_1, \ldots, \xi_{L_1}, x_1, \ldots, x_N)$ and $(y, \zeta) = (y_1, \ldots, y_N, \zeta_1, \ldots, \zeta_{L_2})$. Using (2.47) to evaluate the expectation value in the integrand we get

$$J_N(\xi, \zeta, \eta, \mu) = \frac{1}{\mathbf{Z}_N^{(2)}} \int d\mu(\mathbf{x}, \mathbf{y}) \Delta_N(x) \Delta_N(y) \prod_{a=1}^{N} \frac{\prod_{\alpha=1}^{L_1} (\xi_\alpha - x_a) \prod_{\beta=1}^{L_2} (\zeta_\alpha - y_a)}{\prod_{j=1}^{M_1} (\eta_j - x_a) \prod_{k=1}^{M_2} (\mu_k - y_a)}$$
$$= \mathbf{I}_N(\xi, \zeta, \eta, \mu).$$

Remark 3.2. From (3.9) and (3.12) in the absence of the $F^{(1)}, F^{(2)}, \bar{F}^{(1)}, \bar{F}^{(2)}$ terms, we obtain the representation which we used in [19]

$$\mathbf{Z}_N^{(2)} = (-1)^{\frac{1}{2}N(N+1)} N! \langle N, -N | g | 0, 0 \rangle.$$

Having the representation (3.12) we evaluate $\mathbf{I}_N(\xi, \zeta, \eta, \mu)$ via Wick's theorem. Before we need certain preliminary relations.

4 Biorthogonal polynomials and dressed fermions

In this section we introduce transformations Ω and Q and dressed fermions $d^{(\alpha)}, \bar{d}^{(\alpha)}$ and $b_\alpha, \bar{b}_\alpha$. We re-write the v.e.v. (3.9) in a form suitable for further calculation by means of Wick's theorem.

(a) **Biorthogonal polynomials**.

Consider the sequence of biorthogonal polynomials associated to the measure $d\mu(x, y)$:

$$\int \int P_n(x) S_m(y) d\mu(x, y) = \delta_{n,m}, \quad n, m \geq 0 \tag{4.1}$$

It is convenient to write down the biorthogonal polynomials in the following form

$$P_n(x) = \frac{1}{\sqrt{h_n}} \sum_{m=0}^{n} K_{nm} x^m, \quad S_n(y) = \frac{1}{\sqrt{h_n}} \sum_{m=0}^{n} y^m (\bar{K}^{-1})_{mn}, \quad n \geq 0 \tag{4.2}$$

where K_{nm} and $(\bar{K}^{-1})_{mn}$ are respectively viewed as entries of semi-infinite matrices K and $(\bar{K}^{-1})$. K is a lower triangular matrix and $\bar{K}$ is an upper triangular one, both have units on the diagonal: $K_{nn} = \bar{K}_{nn} = 1$, $n = 0, 1, 2, \ldots$.

As is well-known, the orthogonality relation (4.1) implies

$$H\bar{K} = KB, \tag{4.3}$$

where $H = diag(h_n)$ and B is the bi-moment matrix,

$$B_{nm} = \int\int x^n y^m d\mu(x,y), \quad m,n \geq 0, \tag{4.4}$$

In time (see [11]), (4.3) was made good use to relate matrix models to integrable systems via the factorization method widely used in soliton theory starting from the basic paper [23]. Then K and $\bar{K}$ may be identified with the Mikhailov-Ueno-Takasaki dressing matrices (see [20], [22]), whose rows give rise to a pair the so-called Baker functions, while columns of $K^{-1}$and of $\bar{K}^{-1}$ give rise to a pair of adjoint Baker functions. In case the dressing matrices are semi-infinite, the first of the Baker functions (related to rows of K) and the second of the adjoint Baker function (related to columns $\bar{K}^{-1}$) take the form of quasi-polynomials.

Here we shall use (4.3) differently.

(b) **Canonical transformation Ω.**

First, we introduce

$$\tilde{S}_n(x) := \sum_{m=0}^{+\infty} x^{-m-1}(K^{-1})_{mn}\sqrt{h_n}, \quad \tilde{P}_n(y) := \sum_{m=0}^{+\infty} \bar{K}_{nm} y^{-m-1}\sqrt{h_n}, \quad n \geq 0 \tag{4.5}$$

Properties of $\tilde{S}_n(x)$ and $\tilde{P}_n(y)$ are described in the Appendix C. The main is that they coincide with Hilbert transforms of biorthogonal polynomials (1.8).

Now, let us define matrices $\omega^{(1)}$ and $\omega^{(2)}$ via

$$e^{\omega^{(1)}} = K, \quad e^{\omega^{(2)}} = \bar{K} \tag{4.6}$$

(Respectively, strictly upper and strictly lower) triangular matrices $\omega^{(1),(2)}$ may be defined in a unique way by a recursion procedure.

We introduce

$$\Omega = \exp \sum_{n>m\geq 0} \left(\omega^{(1)}_{n,m} f^{(1)}_n \bar{f}^{(1)}_m + \omega^{(2)}_{m,n} f^{(2)}_{-m-1} \bar{f}^{(2)}_{-n-1} \right) \tag{4.7}$$

Note that, for any N_1 and N_2,

$$\langle N_1, -N_2 | \Omega^{-1} = \langle N_1, -N_2 | \ , \quad \Omega|0,0\rangle = |0,0\rangle \tag{4.8}$$

The first equality in (4.8) follows from (2.39),(2.41) and from the restriction $n > m$ in the summation in (4.7). The second equality in (4.8) follows from the restriction $m \geq 0$ and from (2.39),(2.41).

Consider

$$d^{(\alpha)}(x) = \Omega f^{(\alpha)}(x)\Omega^{-1}, \quad \bar{d}^{(\alpha)}(x) = \Omega \bar{f}^{(\alpha)}(x)\Omega^{-1}, \quad \alpha = 1,2 \tag{4.9}$$

which we refer as dressed fermions.

We have (see Appendix B)

$$d^{(1)}(x) = \sum_{n=-\infty}^{+\infty} f_n^{(1)} P_n(x)\sqrt{h_n} \tag{4.10}$$

$$\bar{d}^{(1)}(x) = \sum_{n=-\infty}^{+\infty} \bar{f}_n^{(1)} \frac{\tilde{S}_n(x)}{\sqrt{h_n}} \tag{4.11}$$

$$d^{(2)}(y) = \sum_{n=-\infty}^{+\infty} f_{-n-1}^{(2)} \frac{\tilde{P}_n(y)}{\sqrt{h_n}} \tag{4.12}$$

$$\bar{d}^{(2)}(y) = \sum_{n=-\infty}^{+\infty} \bar{f}_{-n-1}^{(2)} S_n(y)\sqrt{h_n} \tag{4.13}$$

where for $n < 0$ we use the following notational convention

$$P_n(x) = S_n(x) = x^n, \quad \tilde{S}_n(x) = \tilde{P}_n(x) = x^{-n-1}, \quad h_n = 1, \quad n < 0 \tag{4.14}$$

A kind of dressed fermionic operators (where powers of x were replaced by Baker functions similar to (4.10)-(4.13)), in a different context, were also introduced in [30] where they were called Krichever-Novikov fermions (see Appendix to [30]).

If we write

$$d^{(\alpha)}(x) = \sum_{n=-\infty}^{+\infty} d_n^{(\alpha)} x^n, \quad \bar{d}^{(\alpha)}(x) = \sum_{n=-\infty}^{+\infty} \bar{d}_n^{(\alpha)} x^{-n-1}, \quad \alpha = 1,2$$

where

$$d_n^{(\alpha)} = \Omega f_n^{(\alpha)} \Omega^{-1}, \quad \bar{d}_n^{(\alpha)} = \Omega \bar{f}_n^{(\alpha)} \Omega^{-1},$$

then

$$d_i^{(1)} = \sum_{n \geq i \geq 0} f_n^{(1)} K_{ni}, \quad i \geq 0, \qquad d_i^{(1)} = f_i^{(1)}, \quad i < 0 \tag{4.15}$$

$$\bar{d}_i^{(1)} = \sum_{i \geq n \geq 0} (K^{-1})_{in} \bar{f}_n^{(1)}, \quad i \geq 0, \qquad \bar{d}_i^{(1)} = \bar{f}_i^{(1)}, \quad i < 0 \tag{4.16}$$

$$d^{(2)}_{-i-1} = \sum_{i \geq n \geq 0} f^{(2)}_{-n-1} \bar{K}_{ni}, \quad i \geq 0, \qquad d^{(2)}_{-i-1} = f^{(2)}_{-i-1}, \quad i < 0 \tag{4.17}$$

$$\bar{d}^{(2)}_{-i-1} = \sum_{n \geq i \geq 0} (\bar{K}^{-1})_{in} \bar{f}^{(2)}_{-n-1}, \quad i \geq 0, \qquad \bar{d}^{(2)}_{-i-1} = \bar{f}^{(2)}_{-i-1}, \quad i < 0 \tag{4.18}$$

(c) **Useful formulae for charged vacuum vectors**.

First, we introduce

$$g_n := e^{f^{(1)}_n \bar{f}^{(2)}_{-n-1}} = 1 + f^{(1)}_n \bar{f}^{(2)}_{-n-1}, \tag{4.19}$$

$$e_n := e^{\bar{f}^{(1)}_n f^{(2)}_{-n-1}} = 1 + \bar{f}^{(1)}_n f^{(2)}_{-n-1} \tag{4.20}$$

We see that $g_n, e_n \in \hat{GL}_\infty$.

We shall also consider powers of these operators

$$(g_n)^p = e^{p f^{(1)}_n \bar{f}^{(2)}_{-n-1}} = 1 + p f^{(1)}_n f^{*(2)}_{-n-1}, \tag{4.21}$$

$$(e_n)^p = e^{p \bar{f}^{(1)}_n f^{(2)}_{-n-1}} = 1 + p \bar{f}^{(1)}_n f^{(2)}_{-n-1} \tag{4.22}$$

where $n \in \mathbb{Z}, p \in \mathbb{C}$.

Now, due to

$$\langle 0,0| e_n g_n = \langle 0,0|(1+\bar{f}^{(1)}_n f^{(2)}_{-n-1})(1+f^{(1)}_n \bar{f}^{(2)}_{-n-1}) = \langle 0,0|(1-1+\bar{f}^{(1)}_n f^{(2)}_{-n-1}) = \langle 0,0|\bar{f}^{(1)}_n f^{(2)}_{-n-1},$$

which is true for $n > 0$, and due to (A.1), (A.2), we conclude that for $N > 0$

$$\langle N, -N| = (-1)^{\frac{1}{2}N(N-1)} \langle 0,0| \prod_{n=0}^{N-1} e_n \prod_{n=0}^{N-1} g_n$$

where $\langle N, -N|$ was defined in (2.30)-(2.32).

In the similar way we obtain a representation we shall need later

$$N_2 > N_1 \geq 0 : \quad \langle N_1, -N_2| = (-1)^{\frac{1}{2}N_1(N_1-1)} \langle 0,0| f^{(2)}_{-N_1-1} \cdots f^{(2)}_{-N_2} \prod_{n=0}^{N_1-1} e_n \prod_{n=0}^{N_1-1} g_n \tag{4.23}$$

(d) **Evaluation of $\Omega g|0,0\rangle$**.

Taking into account that $f^{(1)}_{-n-1}|0,0\rangle = \bar{f}^{(2)}_n|0,0\rangle = 0$ for $n > 0$, one obtains

$$\Omega g|0,0\rangle = e^{\sum_{i,j,n,m \geq 0} K_{in} B_{nm} \bar{K}_{mj} f^{(1)}_i \bar{f}^{(2)}_{-j-1}} |0,0\rangle = Q^{-1}|0,0\rangle, \quad Q^{-1} := \prod_{n=0}^{+\infty} (g_n)^{h_n} \tag{4.24}$$

where we used (4.15),(4.18) and (4.3).

(e) **Fermionic operators** $b_\alpha, \bar{b}_\alpha$.

We shall also need the following fermionic operators

$$b_\alpha(x) := Qd^{(\alpha)}(x)Q^{-1} = Q\Omega\, f^{(\alpha)}(Q\Omega)^{-1}, \quad \bar{b}_\alpha(x) := Q\bar{d}^{(\alpha)}(x)Q^{-1} = Q\Omega\, \bar{f}^{(\alpha)}(Q\Omega)^{-1} \tag{4.25}$$

Using (A.3)-(A.6) we write down

$$b_1(\xi) = \sum_{n=-\infty}^{+\infty} f_n^{(1)} P_n(\xi)\sqrt{h_n} = d^{(1)}(\xi) \ , \tag{4.26}$$

$$\bar{b}_1(\eta) = \sum_{n=-\infty}^{+\infty} \bar{f}_n^{(1)} \frac{\tilde{S}_n(\eta)}{\sqrt{h_n}} + \sum_{n=0}^{+\infty} \bar{f}_{-n-1}^{(2)} \tilde{S}_n(\eta)\sqrt{h_n} \ , \tag{4.27}$$

$$b_2(\mu) = \sum_{n=-\infty}^{+\infty} f_{-n-1}^{(2)} \frac{\tilde{P}_n(\mu)}{\sqrt{h_n}} - \sum_{n=0}^{+\infty} f_n^{(1)} \tilde{P}_n(\mu)\sqrt{h_n} \ , \tag{4.28}$$

$$\bar{b}_2(\zeta) = \sum_{n=-\infty}^{+\infty} \bar{f}_{-n-1}^{(2)} S_n(\zeta)\sqrt{h_n} = \bar{d}^{(2)}(\zeta) \tag{4.29}$$

Notice that each of $\bar{b}_1$ and b_2 contains both component fermions.

The fermionic operators $b_\alpha, \bar{b}_\alpha$ may be also called dressed fermions. Similarly to (3.4)-(3.7), for their products, we shall use large letters, namely, we define

$$B_1(\xi) := Q\Omega\, F^{(1)}(\xi)\, (Q\Omega)^{-1}, \quad B_2(\mu) := Q\Omega\, F^{(2)}(\mu)\, (Q\Omega)^{-1},$$
$$\bar{B}_1(\eta) := Q\Omega\, \bar{F}^{(1)}(\eta)\, (Q\Omega)^{-1}, \quad \bar{B}_2(\zeta) := Q\Omega\, \bar{F}^{(2)}(\zeta)\, (Q\Omega)^{-1}$$

(f) $\mathbf{I}_N$ **as a re-written vacuum expectation value**

Now we restate (3.12) as

$$\mathbf{I}_N(\xi,\zeta,\eta,\mu)R_N^{-1} = \langle N_1, -N_2|Q^{-1}\, B_2(\mu)B_1(\xi)\bar{B}_1(\eta)\bar{B}_2(\zeta)|0,0\rangle \tag{4.30}$$

where R_N was defined by (3.10).

Now we have to consider the cases (1),(2),(3) separately.

5 Determinantal expression for integrals of rational symmetric functions

Below we apply Wick's theorem to evaluate v.e.v. in the right hand side of (4.30), getting answers for $\mathbf{I}_N^{(2)}(\xi,\zeta,\eta,\mu)$ for all the cases listed in the Introduction.

5.1 Determinantal representation in the case (1): $N_2 \geq N_1 \geq 0$

In the case (1) we have the following formulae for $\langle N_1, -N_2|Q^{-1}$:

$$N_2 \geq N_1 \geq 0: \quad \langle N_1, -N_2|Q^{-1} = (-1)^{\frac{1}{2}N_1(N_1+1)}\langle 0,0|f^{(2)}_{-N_1-1}\cdots f^{(2)}_{-N_2} E \prod_{n=0}^{N_1-1} h_n \ , \qquad (5.1)$$

where

$$E := \prod_{n=0}^{N_1-1} (e_n)^{-h_n^{-1}} \qquad (5.2)$$

Formula (5.1) follows from relations

$$\langle 0,0|e_n g_n^{1+h_n} = \langle 0,0|(1+\bar{f}^{(1)}_n f^{(2)}_{-n-1})(1+(1+h_n)f^{(1)}_n \bar{f}^{(2)}_{-n-1}) = \langle 0,0|(-h_n + \bar{f}^{(1)}_n f^{(2)}_{-n-1})$$

$$= -h_n \langle 0,0|e_n^{-h_N^{-1}}, \quad n \geq 0$$

and from (4.23).

It is important that

$$E|0,0\rangle = 0 \ , \qquad (5.3)$$

which is true as each $e_n^p|0,0\rangle = 0, n \geq 0$.

Using (A.3)-(A.6), we may evaluate

$$a_1(\xi) := Eb_1(\xi)E^{-1} = \sum_{n=-\infty}^{+\infty} f^{(1)}_n P_n(\xi)\sqrt{h_n} + \sum_{n=0}^{N_1-1} f^{(2)}_{-n-1}\frac{P_n(\xi)}{\sqrt{h_n}} \ , \qquad (5.4)$$

$$\bar{a}_1(\eta) := E\bar{b}_1(\eta)E^{-1} = \sum_{n=N_1}^{+\infty} \bar{f}^{(1)}_n \frac{\tilde{S}_n(\eta)}{\sqrt{h_n}} + \sum_{n=-\infty}^{-1} \bar{f}^{(1)}_n \frac{\tilde{S}_n(\eta)}{\sqrt{h_n}} + \sum_{n=0}^{+\infty} \bar{f}^{(2)}_{-n-1}\tilde{S}_n(\eta)\sqrt{h_n} \ , \qquad (5.5)$$

$$a_2(\mu) := Eb^{(2)}(\mu)E^{-1} = \sum_{n=N_1}^{+\infty} f^{(2)}_{-n-1}\frac{\tilde{P}_n(\mu)}{\sqrt{h_n}} + \sum_{n=-\infty}^{-1} f^{(2)}_{-n-1}\frac{\tilde{P}_n(\mu)}{\sqrt{h_n}} - \sum_{n=0}^{+\infty} f^{(1)}_n \tilde{P}_n(\mu)\sqrt{h_n} \ , \qquad (5.6)$$

$$\bar{a}_2(\zeta) := E\bar{b}^{(2)}(\zeta)E^{-1} = \sum_{n=-\infty}^{+\infty} \bar{f}^{(2)}_{-n-1} S_n(\zeta)\sqrt{h_n} - \sum_{n=0}^{N_1-1} \bar{f}^{(1)}_n \frac{S_n(\zeta)}{\sqrt{h_n}} \qquad (5.7)$$

Using notations

$$A_1(\xi) := EQ\Omega\ F^{(1)}(\xi)\ (EQ\Omega)^{-1}, \quad A_2(\mu) := EQ\Omega\ F^{(2)}(\mu)\ (EQ\Omega)^{-1},$$

$$\bar{A}_1(\eta) := EQ\Omega\ \bar{F}^{(1)}(\eta)\ (EQ\Omega)^{-1}, \quad \bar{A}_2(\zeta) := EQ\Omega\ \bar{F}^{(2)}(\zeta)\ (EQ\Omega)^{-1}$$

for the products, by analogy with (3.4)-(3.7), by (5.1) and (5.3), we arrive at

$$\mathbf{I}_N(\xi,\zeta,\eta,\mu)R_N^{-1} = (-1)^{\frac{1}{2}N_1(N_1+1)}\prod_{n=0}^{N_1-1} h_n\langle 0,0|f^{(2)}_{-N_1-1}\cdots f^{(2)}_{-N_2}A_2(\mu)A_1(\xi)\bar{A}_1(\eta)\bar{A}_2(\zeta)|0,0\rangle \tag{5.8}$$

Finely, this form is applicable to apply the Wick's formula (2.8). Indeed, each a_i is a linear combination of fermions $f^{(1)}$ and $f^{(2)}$, while each $\bar{a}_i$ is a linear combination of fermions $\bar{f}^{(1)}$ and $\bar{f}^{(2)}$.

Then, by (2.8), the vacuum expectation value in formula (5.8) is equal to the determinant of a L_1+M_2 by L_1+M_2 matrix, which consists of six blocks formed by pair-wise v.e.v.:

$$\begin{pmatrix} \langle a_1(\xi_\alpha)\bar{a}_1(\eta_j)\rangle & \langle a_2(\mu_k)\bar{a}_1(\eta_j)\rangle & \langle f^{(2)}_{i-1-N-L_2+M_2}\bar{a}_1(\eta_j)\rangle \\ \langle a_1(\xi_\alpha)\bar{a}_2(\zeta_\beta)\rangle & \langle a_2(\mu_k)\bar{a}_2(\zeta_\beta)\rangle & \langle f^{(2)}_{i-1-N-L_2+M_2}\bar{a}_2(\zeta_\beta)\rangle \end{pmatrix}$$

where

$$\alpha = 1,\ldots,L_1; \quad \beta = 1,\ldots,L_2; \quad j = 1,\ldots,M_1; \quad k = 1,\ldots,M_2$$

$$i = 1,\ldots,M_1+L_2-L_1-M_2$$

Now, taking into account (2.29)-(2.29), from the explicit formulae (5.4)-(5.7), and also from (C.3) and (C.4), we compute all relevant pair-wise vacuum expectation values

$$\langle a_1(\xi_i)\bar{a}_1(\eta_j)\rangle = \frac{1}{\xi_i-\eta_j} + \sum_{n=0}^{N_1-1} P_n(\xi_i)\tilde{S}_n(\eta_j),$$

$$\langle a_2(\mu_i)\bar{a}_2(\zeta_j)\rangle = \sum_{n=N_1}^{+\infty} S_n(\zeta_j)\tilde{P}_n(\mu_i) = -\frac{1}{\zeta_j-\mu_i} - \sum_{n=0}^{N_1-1} S_n(\zeta_j)\tilde{P}_n(\mu_i),$$

$$\langle a_1(\xi_i)\bar{a}_2(\zeta_j)\rangle = \sum_{n=0}^{N_1-1} P_n(\xi_i)S_n(\zeta_j),$$

$$\langle a_2(\mu_i)\bar{a}_1(\eta_j)\rangle = \sum_{n=N_1}^{+\infty} \tilde{S}_n(\eta_j)\tilde{P}_n(\mu_i) = \int\int\frac{d\mu(x,y)}{(\eta_j-x)(\mu_i-y)} - \sum_{n=0}^{N_1-1}\tilde{S}_n(\eta_j)\tilde{P}_n(\mu_i),$$

$$\langle f^{(2)}_{i-1-N-L_2+M_2}\bar{a}_1(\eta_j)\rangle = \sqrt{h_{N+L_2-M_2-i}}\tilde{S}_{N+L_2-M_2-i}(\eta_j),$$

$$\langle f^{(2)}_{i-1-N-L_2+M_2}\bar{a}_2(\zeta_j)\rangle = \sqrt{h_{N+L_2-M_2-i}}S_{N+L_2-M_2-i}(\zeta_j)$$

After trivial manipulations with rows and columns of the matrix of pair-wise v.e.v. we obtain the answer:

$$\begin{aligned}\mathbf{I}_N(\xi,\zeta,\eta,\mu) &= (-1)^{\frac{1}{2}(M_1+M_2)(M_1+M_2-1)}(-1)^{L_2M_1}\prod_{n=N}^{N+L_1-M_1-1}\sqrt{h_n}\prod_{n=N}^{N+L_2-M_2-1}\sqrt{h_n}\\ &\times\frac{\prod_{\alpha=1}^{L_1}\prod_{j=1}^{M_1}(\xi_\alpha-\eta_j)\prod_{\beta=1}^{L_2}\prod_{k=1}^{M_2}(\zeta_\beta-\mu_k)}{\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)}\,\mathrm{det}G,\end{aligned} \tag{5.9}$$

where the matrix G is $(L_2+M_1)\times(L_2+M_1)$ matrix which consists of six blocks:

$$\begin{pmatrix} \overset{N_1}{K}_{11}(\mu_k,\zeta_\beta) & \overset{N_1}{K}_{21}(\xi_\alpha,\zeta_\beta) & S_{N+L_1-M_1}(\zeta_\beta) & \dots & S_{N+L_2-M_2-1}(\zeta_\beta)\\ \overset{N_1}{K}_{12}(\mu_k,\eta_j) & \overset{N_1}{K}_{22}(\xi_\alpha,\eta_j) & \tilde{S}_{N+L_1-M_1}(\eta_j) & \dots & \tilde{S}_{N+L_2-M_2-1}(\eta_j)\end{pmatrix}$$

where

$$\alpha=1,\dots,L_1;\quad \beta=1,\dots,L_2;\quad j=1,\dots,M_1;\quad k=1,\dots,M_2$$

and where $N_1=N+L_1-M_1$ and

$$\overset{J}{K}_{11}(\mu,\zeta)=\sum_{n=0}^{J-1}\tilde{P}_n(\mu)S_n(\zeta)+\frac{1}{\zeta-\mu} \tag{5.10}$$

$$\overset{J}{K}_{22}(\xi,\eta)=\sum_{n=0}^{J-1}P_n(\xi)\tilde{S}_n(\eta)+\frac{1}{\xi-\eta} \tag{5.11}$$

$$\overset{J}{K}_{21}(\xi,\zeta)=\sum_{n=0}^{J-1}P_n(\xi)S_n(\zeta) \tag{5.12}$$

$$\overset{J}{K}_{12}(\mu,\eta)=\sum_{n=0}^{J-1}\tilde{P}_n(\mu)\tilde{S}_n(\eta)-\int\frac{d\mu(x,y)}{(\eta-x)(\mu-y)} \tag{5.13}$$

$$S_{N+L_1-M_1+i-1}(\zeta_j),\quad i=1,\dots,M_1-L_1-M_2+L_2,\quad j=1,\dots,L_2 \tag{5.14}$$

$$\tilde{S}_{N+L_1-M_1+i-1}(\eta_j),\quad i=1,\dots,M_1-L_1-M_2+L_2,\quad j=1,\dots,M_1, \tag{5.15}$$

Now we shall consider six examples, related to the six-block structure, where, in each case, only one entry contributes.

Example 1. $M_1 = 1$ and $L_1 = L_2 = M_2 = 0$, thus $N_2 > N_1 \geq 0$. We put $\eta_1 = \eta$. In this case the matrix has only one non-vanishing element, giving the well-known formula

$$\mathbf{I}_N(\eta) = \frac{1}{\sqrt{h_{N-1}}} \tilde{S}_{N-1}(\eta) \tag{5.16}$$

Example 2. $L_2 = 1$ and $L_1 = M_1 = M_2 = 0$, thus $N_2 > N_1 \geq 0$. We put $\zeta_1 = \zeta$. In this case the matrix has only one non-vanishing element, giving the well-known formula

$$\mathbf{I}_N(\zeta) = \sqrt{h_N} S_N(\zeta) \tag{5.17}$$

Examples below are related to the equality $N_2 = N_1 \geq 0$.

Example 3. $M_1 = L_1 = 1$ and $L_2 = M_2 = 0$. We put $\xi_1 = \xi$ and $\eta_1 = \eta$. In this case the matrix has only one non-vanishing element and we obtain

$$\mathbf{I}_N(\xi, \eta) = 1 + (\xi - \eta) \sum_{n=0}^{N-1} P_n(\xi) \tilde{S}_n(\eta) \tag{5.18}$$

Similarly, we have

Example 4. $L_2 = M_2 = 1$ and $L_1 = M_1 = 0$. In this case the matrix has only one non-vanishing element, and we obtain

$$\mathbf{I}_N(\zeta, \mu) = -(\zeta - \mu) \sum_{n=N}^{+\infty} S_n(\zeta) \tilde{P}_n(\mu) = 1 + (\zeta - \mu) \sum_{n=0}^{N-1} S_n(\zeta) \tilde{P}_n(\mu) \tag{5.19}$$

Example 5. $M_1 = M_2 = 1$ and $L_1 = L_2 = 0$. In this case the matrix has only one non-vanishing element. We obtain

$$\mathbf{I}_N(\eta, \mu) = \frac{1}{h_{N-1}} \sum_{n=N-1}^{+\infty} \tilde{S}_n(\eta) \tilde{P}_n(\mu) = \frac{1}{h_{N-1}} \int\int \frac{d\mu(x,y)}{(\eta - x)(\mu - y)} - \frac{1}{h_{N-1}} \sum_{n=0}^{N-2} \tilde{S}_n(\eta) \tilde{P}_n(\mu) \tag{5.20}$$

Example 6. $L_1 = L_2 = 1$ and $M_1 = M_2 = 0$. In this case the matrix has only one non-vanishing element and we obtain

$$\mathbf{I}_N(\xi, \zeta) = h_N \sum_{n=0}^{N} P_n(\xi) S_n(\zeta) \tag{5.21}$$

Evaluation for the case $N_1 \geq N_2 \geq 0$

This case may be obtained from the previous one, by interchanging subscripts $L_1 \leftrightarrow L_2$, $M_1 \leftrightarrow M_2$, $N_1 \leftrightarrow N_2$, and $\xi \leftrightarrow \zeta$, $\mu \leftrightarrow \eta$ and also $P_n \leftrightarrow S_n$, $\tilde{P}_n \leftrightarrow \tilde{S}_n$.

We obtain the answer which coincides with the answer given by formulae (1.8)-(1.16) of [1]:

$$\begin{aligned}\mathbf{I}_N(\xi,\zeta,\eta,\mu) &= (-1)^{\frac{1}{2}(M_1+M_2)(M_1+M_2-1)}(-1)^{L_1M_2}\prod_{n=0}^{N-1}h_n^{-1}\prod_{n=N}^{N+L_2-M_2-1}\sqrt{h_n}\prod_{n=N}^{N+L_1-M_1-1}\sqrt{h_n}\\ &\times\frac{\prod_{\alpha=1}^{L_1}\prod_{j=1}^{M_1}(\xi_\alpha-\eta_j)\prod_{\beta=1}^{L_2}\prod_{k=1}^{M_2}(\zeta_\beta-\mu_k)}{\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)}\ \det G,\end{aligned} \tag{5.22}$$

where $(L_1+M_2)\times(L_1+M_2)$ matrix G consists of six blocks:

$$\begin{pmatrix} \overset{N_2}{K}_{11}(\xi_\alpha,\eta_j) & \overset{N_2}{K}_{12}(\xi_\alpha,\zeta_\beta) & P_{N+L_2-M_2}(\xi_\alpha) & \dots & P_{N+L_1-M_1-1}(\xi_\alpha) \\ \overset{N_2}{K}_{21}(\mu_k,\eta_j) & \overset{N_2}{K}_{22}(\mu_k,\zeta_\beta) & \tilde{P}_{N+L_2-M_2}(\mu_k) & \dots & \tilde{P}_{N+L_1-M_1-1}(\mu_k) \end{pmatrix}$$

where

$$\alpha=1,\dots,L_1;\quad \beta=1,\dots,L_2;;\quad j=1,\dots,M_1;\quad k=1,\dots,M_2$$

and where $N_2=N+L_2-M_2$ and

$$\overset{J}{K}_{11}(\xi,\eta)=\sum_{n=0}^{J-1}P_n(\xi)\tilde{S}_n(\eta)+\frac{1}{\xi-\eta} \tag{5.23}$$

$$\overset{J}{K}_{22}(\mu,\zeta)=\sum_{n=0}^{J-1}\tilde{P}_n(\mu)S_n(\zeta)+\frac{1}{\zeta-\mu} \tag{5.24}$$

$$\overset{J}{K}_{12}(\xi,\zeta)=\sum_{n=0}^{J-1}P_n(\xi)S_n(\zeta) \tag{5.25}$$

$$\overset{J}{K}_{21}(\mu,\eta)=\sum_{n=0}^{J-1}\tilde{P}_n(\mu)\tilde{S}_n(\eta)-\int\frac{d\mu(x,y)}{(\eta-x)(\mu-y)} \tag{5.26}$$

Example 7. $M_2=1$ and $L_2=L_1=M_1=0$, thus $N_1>N_2\geq 0$. We put $\mu_1=\mu$. Then

$$\mathbf{I}_N(\mu)=\frac{1}{\sqrt{h_{N-1}}}\tilde{P}_{N-1}(\mu) \tag{5.27}$$

Example 8. $L_1=1$ and $L_2=M_2=M_1=0$, thus $N_1>N_2\geq 0$. We put $\xi_1=\xi$. We obtain the well-known formula

$$\mathbf{I}_N(\xi)=\sqrt{h_N}P_N(\xi) \tag{5.28}$$

5.2 When $N_1 < 0$

In cases listed in the Introduction as (2) and (3) we have $N_1 < 0$. Let us explicitly write down

$$N_1 \le 0 \le N_2 : \quad \langle N_1, -N_2| = \langle 0,0| f^{(1)}_{-1} \cdots f^{(1)}_{N_1} f^{(2)}_{-1} \cdots f^{(2)}_{-N_2} \ , \tag{5.29}$$

$$N_1 \le N_2 \le 0 : \quad \langle N_1, -N_2| = \langle 0,0| f^{(1)}_{-1} \cdots f^{(1)}_{N_1} \bar{f}^{(2)}_{0} \cdots \bar{f}^{(2)}_{-N_2-1} \tag{5.30}$$

Then, as it follows from (A.3)-(A.4), in either case

$$\langle N_1, -N_2| Q^{-1} = \langle N_1, -N_2| \tag{5.31}$$

Thus, in the both cases, we re-write (4.30) as

$$\mathbf{I}_N(\xi,\zeta,\eta,\mu) R_N^{-1} = \langle N_1, -N_2| B_2(\mu) B_1(\xi) \bar{B}_1(\eta) \bar{B}_2(\zeta) |0,0\rangle \tag{5.32}$$

5.3 Evaluation for the case (2): $N_1 \le 0 \le N_2$.

Thus, by (5.32) and (5.29) we have

$$\mathbf{I}_N(\xi,\zeta,\eta,\mu) R_N^{-1} = \langle 0,0| f^{(1)}_{-1} \cdots f^{(1)}_{N_1} f^{(2)}_{-1} \cdots f^{(2)}_{-N_2} B_2(\mu) B_1(\xi) \bar{B}_1(\eta) \bar{B}_2(\zeta) |0,0\rangle \tag{5.33}$$

By Wick's formula (2.8) the right hand side is the determinant of a L_2+M_1 by L_2+M_1 matrix which consists of eight blocks:

$$\begin{pmatrix} \langle b_1(\xi_\alpha)\bar{b}_1(\eta_j)\rangle & \langle b_2(\mu_k)\bar{b}_1(\eta_j)\rangle & \langle f^{(2)}_{b-1-N-L_2+M_2}\bar{b}_1(\eta_j)\rangle & \langle f^{(1)}_{m-1+N+L_1-M_1}\bar{b}_1(\eta_j)\rangle \\ \langle b_1(\xi_\alpha)\bar{b}_2(\zeta_\beta)\rangle & \langle b_2(\mu_k)\bar{b}_2(\zeta_\beta)\rangle & \langle f^{(2)}_{b-1-N-L_2+M_2}\bar{b}_2(\zeta_\beta)\rangle & \langle f^{(1)}_{m-1+N+L_1-M_1}\bar{b}_2(\zeta_\beta)\rangle \end{pmatrix}$$

where

$$\alpha = 1,\ldots,L_1; \quad \beta = 1,\ldots,L_2; \quad j = 1,\ldots,M_1; \quad k = 1,\ldots,M_2;$$
$$b = 1,\ldots,N+L_2-M_2; \quad m = 1,\ldots,-N-L_1+M_1$$

Then using (4.26)-(4.29), we evaluate all pair-wise vacuum expectation values:

$$\langle b_1(\xi_\alpha)\bar{b}_1(\eta_j)\rangle = \frac{1}{\xi_\alpha}\frac{1}{1-\frac{\eta_j}{\xi_\alpha}} = \frac{1}{\xi_\alpha - \eta_j},$$

$$\langle b_2(\mu_k)\bar{b}_2(\zeta_\beta)\rangle = \sum_{n=0}^{+\infty} \tilde{P}_n(\mu_k) S_n(\zeta_\beta) = \frac{1}{\mu_k - \zeta_\beta},$$

$$\langle b_2(\mu_k)\bar{b}_1(\eta_j)\rangle = \sum_{n=0}^{+\infty} \tilde{P}_n(\mu_k)\tilde{S}_n(\eta_j) = H(\mu_k,\eta_j)$$

$$\langle b_1(\xi_\alpha)\bar{b}_2(\zeta_\beta)\rangle = 0,$$

$$\langle f^{(2)}_{b-1-N-L_2+M_2}\bar{b}_1(\eta_j)\rangle = \sqrt{h_{N+L_2-M_2-b}}\tilde{S}_{N+L_2-M_2-b}(\eta_j), \quad b=1,\dots,N+L_2-M_2,$$

$$\langle f^{(2)}_{b-1-N-L_2+M_2}\bar{b}_2(\zeta_\beta)\rangle = \sqrt{h_{N+L_2-M_2-b}}S_{N+L_2-M_2-b}(\zeta_\beta), \quad b=1,\dots,N+L_2-M_2,$$

$$\langle f^{(1)}_{m-1+N+L_1-M_1}\bar{b}_1(\eta_j)\rangle = \eta_j^{-N-L_1+M_1-m}, \quad m=1,\dots,-N-L_1+M_1,$$

$$\langle f^{(1)}_{m-1+N+L_1-M_1}\bar{b}_2(\zeta_\beta)\rangle = 0, \quad m=1,\dots,-N-L_1+M_1$$

After trivial manipulations with rows and columns of the matrix of pair-wise v.e.v. we obtain the answer

$$\mathbf{I}_N(\xi,\zeta,\eta,\mu) =$$

$$= \epsilon \prod_{n=N}^{N+L_1-M_1-1}\sqrt{h_n}\prod_{n=N}^{N+L_2-M_2-1}\sqrt{h_n}\frac{\prod_{\alpha=1}^{L_1}\prod_{j=1}^{M_1}(\xi_\alpha-\eta_j)\prod_{\beta=1}^{L_2}\prod_{k=1}^{M_2}(\zeta_\beta-\mu_k)}{\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)}\mathrm{det}G, \tag{5.34}$$

$$\epsilon = (-1)^{\frac{1}{2}N(N-1)+N(L_1-M_1)+\frac{1}{2}M_1(M_1-1)+\frac{1}{2}L_2(L_2-1)+L_1(L_2-M_2)+L_2M_2-M_2}$$

where the $(L_2+M_1)\times(L_2+M_1)$ matrix G consists of eight blocks

$$\begin{pmatrix} \frac{1}{\mu_k-\zeta_\beta} & 0 & S_b(\zeta_\beta) & 0 \\ H(\mu_k,\eta_j) & \frac{1}{\xi_\alpha-\eta_j} & \tilde{S}_b(\eta_j) & S_m(\eta_j) \end{pmatrix} \tag{5.35}$$

where

$$\alpha=1,\dots,L_1; \quad \beta=1,\dots,L_2; \quad j=1,\dots,M_1; \quad k=1,\dots,M_2;$$

$$b=1,\dots,N+L_2-M_2; \quad m=1,\dots,-N-L_1+M_1$$

Evaluation for the case $N_2 \le 0 \le N_1$.

The answer for this case may be obtained by the answer for the previous case, if we inter-change subscripts $L_1 \leftrightarrow L_2$, $M_1 \leftrightarrow M_2$, $N_1 \leftrightarrow N_2$, and the variables $\xi \leftrightarrow \zeta$, $\mu \leftrightarrow \eta$ and also $P_n \leftrightarrow S_n$, $\tilde{P}_n \leftrightarrow \tilde{S}_n$, and (due to the definition (C.4)) $H(\mu,\eta) \to H(\mu,\eta)$.

The answer coincides with the answer of [1] (see (3.21)-(3.22) there):

$$\mathbf{I}_N(\xi,\zeta,\eta,\mu) =$$

$$= \epsilon \prod_{n=N}^{N+L_2-M_2-1}\sqrt{h_n}\prod_{n=N}^{N+L_1-M_1-1}\sqrt{h_n}\frac{\prod_{\alpha=1}^{L_1}\prod_{j=1}^{M_1}(\xi_\alpha-\eta_j)\prod_{\beta=1}^{L_2}\prod_{k=1}^{M_2}(\zeta_\beta-\mu_k)}{\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)}\mathrm{det}G, \tag{5.36}$$

$$\epsilon = (-1)^{\frac{1}{2}N(N-1)+N(L_2-M_2)+\frac{1}{2}M_2(M_2-1)+\frac{1}{2}L_1(L_1-1)+L_2(L_1-M_1)+L_1M_1-M_1}$$

where the $(L_1+M_2)\times(L_1+M_2)$ matrix G consists of eight blocks

$$\begin{pmatrix} \frac{1}{\xi_\alpha-\eta_i} & 0 & P_b(\xi_\alpha) & 0 \\ H(\mu_k,\eta_i) & \frac{1}{\mu_k-\zeta_\beta} & \tilde{P}_b(\mu_k) & S_m(\mu_k) \end{pmatrix} \tag{5.37}$$

where

$$\alpha=1,\dots,L_1;\quad \beta=1,\dots,L_2;\quad i=1,\dots,M_1;\quad k=1,\dots,M_2;$$
$$b=1,\dots,N+L_1-M_1;\quad m=1,\dots,M_2-L_2-N$$

5.4 Evaluation for the case (3): $N_1\le N_2\le 0$

In this case we have (5.32).

Using transformation (2.42)-(2.43) which leaves vacuum expectation values invariant, in form:

$$\langle *,*| \to \langle *,*+N_2|,\quad |*,*\rangle \to |*,*+N_2\rangle,$$
$$f_i^{(2)} \to f_{i+N_2}^{(2)},\quad \bar{f}_i^{(2)} \to \bar{f}_{i+N_2}^{(2)},\quad i\in\mathbb{Z},$$

we write

$$\begin{aligned}\mathbf{I}_N(\xi,\zeta,\eta,\mu)R_N^{-1} &= \langle N_1,-N_2|B_2(\mu)B_1(\xi)\bar{B}_1(\eta)\bar{B}_2(\zeta)|0,0\rangle \\ &= \langle N_1,0|C_2(\mu)C_1(\xi)\bar{C}_1(\eta)\bar{C}_2(\zeta)|0,N_2\rangle\end{aligned} \tag{5.38}$$

where we shift the charges of the second component of the vacuum vector by N_2 and replaced fermionic operators $b_1(\xi),\bar{b}_1(\eta),b_2(\mu)$ and $\bar{b}_2(\zeta)$ given by (4.26)-(4.29), by

$$c_1(\xi)=\sum_{n=-\infty}^{+\infty} f_n^{(1)}P_n(\xi)\sqrt{h_n}\ , \tag{5.39}$$

$$c_1(\eta)=\sum_{n=-\infty}^{+\infty} \bar{f}_n^{(1)}\frac{\tilde{S}_n(\eta)}{\sqrt{h_n}}+\sum_{n=0}^{+\infty}\bar{f}_{-n-1+N_2}^{(2)}\tilde{S}_n(\eta)\sqrt{h_n}\ , \tag{5.40}$$

$$c_2(\mu)=\sum_{n=-\infty}^{+\infty} f_{-n-1+N_2}^{(2)}\frac{\tilde{P}_n(\mu)}{\sqrt{h_n}}-\sum_{n=0}^{+\infty} f_n^{(1)}\tilde{P}_n(\mu)\sqrt{h_n}\ , \tag{5.41}$$

$$\bar{c}_2(\zeta)=\sum_{n=-\infty}^{+\infty}\bar{f}_{-n-1+N_2}^{(2)}S_n(\zeta)\sqrt{h_n} \tag{5.42}$$

We produced this shift of the vacuum charge and the fermion numbering in order to come to the form of v.e.v., where all fermions with bar are situated to the right:

$$\langle 0,0|f^{(1)}_{-1}\cdots f^{(1)}_{N_1}\prod_{n=1}^{M_2}c_2(\mu_n)\prod_{n=1}^{L_1}c_1(\xi_n)\prod_{n=1}^{M_1}\bar{c}_1(\eta_n)\prod_{n=1}^{L_2}\bar{c}_2(\zeta_n)\bar{f}^{(2)}_{N_2}\cdots\bar{f}^{(2)}_{-1}|0,0\rangle, \tag{5.43}$$

where formula (2.8) is applicable. We have nine-block matrix:

$$\begin{pmatrix} \langle c_1(\xi_\alpha)\bar{c}_1(\eta_j)\rangle & \langle c_1(\xi_\alpha)\bar{c}_2(\zeta_\beta)\rangle & \langle c_1(\xi_\alpha)\bar{f}^{(2)}_{m-1+N+L_2-M_2}\rangle \\ \langle c_2(\mu_k)\bar{c}_1(\eta_j)\rangle & \langle c_2(\mu_k)\bar{c}_2(\zeta_\beta)\rangle & \langle c_2(\mu_k)\bar{f}^{(2)}_{m-1+N+L_2-M_2}\rangle \\ \langle f^{(1)}_{\ell-1+N+L_1-M_1}\bar{c}_1(\eta_j)\rangle & \langle f^{(1)}_{\ell-1+N+L_1-M_1}\bar{c}_2(\zeta_\beta)\rangle & 0 \end{pmatrix}$$

where

$$\alpha=1,\ldots,L_1;\quad \beta=1,\ldots,L_2;\quad j=1,\ldots,M_1;\quad k=1,\ldots,M_2;$$

$$\ell=1,\ldots,-N-L_1+M_1;\quad m=1,\ldots,N+L_2-M_2$$

Pair-wise expectation values are

$$\langle c_1(\xi)\bar{c}_1(\eta)\rangle=\frac{1}{\xi}\frac{1}{1-\frac{\eta}{\xi}}=\frac{1}{\xi-\eta},$$

$$\langle c_2(\mu)\bar{c}_2(\zeta)\rangle=\sum_{n=0}^{+\infty}\tilde{P}_n(\mu)S_n(\zeta)=\frac{1}{\mu-\zeta},$$

$$\langle c_2(\mu)\bar{c}_1(\eta)\rangle=\sum_{n=0}^{+\infty}\tilde{P}_n(\mu)\tilde{S}_n(\eta)=H(\mu,\eta),$$

$$\langle c_1(\xi)\bar{c}_2(\zeta)\rangle=0,$$

$$\langle f^{(1)}_{\ell-1+N+L_1-M_1}\bar{c}_1(\eta)\rangle=\tilde{S}_{\ell-1+N+L_1-M_1}(\eta)\sqrt{h_{\ell-1+N+L_1-M_1}}=\eta^{-N-L_1+M_1-\ell},\quad \ell=1,\ldots,-N-L_1+M_1,$$

$$\langle f^{(1)}_{\ell-1+N+L_1-M_1}\bar{c}_2(\zeta)\rangle=0,\quad \ell=1,\ldots,-N-L_1+M_1,$$

$$\langle c_1(\xi)\bar{f}^{(2)}_{m-1+N+L_2-M_2}\rangle=0,\quad m=1,\ldots,N+L_2-M_2,$$

$$\langle c_2(\mu)\bar{f}^{(2)}_{m-1+N+L_2-M_2}\rangle=\tilde{P}_{-m}(\mu)\sqrt{h_{-m}}=\mu^{\ell-1},\quad m=1,\ldots,N+L_2-M_2$$

After trivial manipulations with rows and columns of the matrix of pair-wise v.e.v. we obtain the answer which coincides with the answer of [1] (given by (3.35)-(3.36) there):

$$\mathbf{I}_N(\xi,\zeta,\eta,\mu)=$$

$$= \epsilon \prod_{n=N}^{N+L_2-M_2-1} \sqrt{h_n} \prod_{n=N}^{N+L_1-M_1-1} \sqrt{h_n} \frac{\prod_{\alpha=1}^{L_1} \prod_{j=1}^{M_1} (\xi_\alpha - \eta_j) \prod_{\beta=1}^{L_2} \prod_{k=1}^{M_2} (\zeta_\beta - \mu_k)}{\Delta_{L_1}(\xi) \Delta_{L_2}(\zeta) \Delta_{M_1}(\eta) \Delta_{M_2}(\mu)} \det G, \qquad (5.44)$$

$$\epsilon = (-1)^{\frac{1}{2}L_2(L_2-1)+\frac{1}{2}M_1(M_1-1)+\frac{1}{2}N(N+1)+L_2(L_1+M_1)+N(L_1+M_1)+M_2L_2}$$

where G is $(M_1 + M_2 - N) \times (M_1 + M_2 - N)$ matrix which consists of nine blocks:

$$\begin{pmatrix} \frac{1}{\xi_\alpha - \eta_j} & 0 & 0 \\ H(\mu_k, \eta_j) & \frac{1}{\mu_k - \zeta_\beta} & S_m(\mu_k) \\ P_\ell(\eta_j) & 0 & 0 \end{pmatrix} \qquad (5.45)$$

where

$$\alpha = 1, \ldots, L_1; \quad \beta = 1, \ldots, L_2; \quad j = 1, \ldots, M_1; \quad k = 1, \ldots, M_2;$$
$$\ell = 1, \ldots, M_1 - L_1 - N; \quad m = 1, \ldots, N + L_2 - M_2$$

Evaluation for the case $N_2 \le N_1 \le 0$

After replacements we obtain

$$\mathbf{I}_N(\xi, \zeta, \eta, \mu) =$$
$$= \epsilon \prod_{n=N}^{N+L_2-M_2-1} \sqrt{h_n} \prod_{n=N}^{N+L_1-M_1-1} \sqrt{h_n} \frac{\prod_{\alpha=1}^{L_1} \prod_{j=1}^{M_1} (\xi_\alpha - \eta_j) \prod_{\beta=1}^{L_2} \prod_{k=1}^{M_2} (\zeta_\beta - \mu_k)}{\Delta_{L_1}(\xi) \Delta_{L_2}(\zeta) \Delta_{M_1}(\eta) \Delta_{M_2}(\mu)} \det G, \qquad (5.46)$$

$$\epsilon = (-1)^{\frac{1}{2}L_1(L_1-1)+\frac{1}{2}M_2(M_2-1)+\frac{1}{2}N(N+1)+L_1(L_2+M_2)+N(L_2+M_2)+M_1L_1}$$

where G is $(M_1 + M_2 - N) \times (M_1 + M_2 - N)$ matrix which consists of nine blocks

$$\begin{pmatrix} \frac{1}{\zeta_\beta - \mu_k} & 0 & 0 \\ H(\mu_k, \eta_j) & \frac{1}{\eta_j - \xi_\alpha} & P_m(\eta_j) \\ S_\ell(\mu_k) & 0 & 0 \end{pmatrix} \qquad (5.47)$$

where

$$\alpha = 1, \ldots, L_1; \quad \beta = 1, \ldots, L_2; \quad j = 1, \ldots, M_1; \quad k = 1, \ldots, M_2;$$
$$\ell = 1, \ldots, M_2 - L_2 - N; \quad m = 1, \ldots, M_1 - L_1 - N$$

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A Appendix A

By (2.24) we have for $p, q \in \mathbb{C}$:

$$[(g_n)^p, (g_m)^q] = 0, \quad [(e_n)^p, (e_m)^q] = 0, \quad n, m \in \mathbb{Z} \tag{A.1}$$

and

$$[(g_n)^p, (e_m)^q] = 0, \quad n \neq m \tag{A.2}$$

where $[,]$ denotes the commutator.

Also:

$$[(g_n)^p, f_m^{(1)}] = [(g_n)^p, \bar{f}_m^{(2)}] = [(e_n)^p, \bar{f}_m^{(1)}] = [(e_n)^p, f_m^{(2)}] = 0, \quad n, m \in \mathbb{Z}, \tag{A.3}$$

$$[(g_n)^p, \bar{f}_m^{(1)}] = [(g_n)^p, f_{-m-1}^{(2)}] = [(e_n)^p, f_m^{(1)}] = [(e_n)^p, \bar{f}_{-m-1}^{(2)}] = 0, \quad n \neq m, \tag{A.4}$$

and

$$(g_n)^{-p} \bar{f}_n^{(1)} (g_n)^p = \bar{f}_n^{(1)} + p \bar{f}_{-n-1}^{(2)}, \quad (g_n)^{-p} f_{-n-1}^{(2)} (g_n)^p = f_{-n-1}^{(2)} - p f_n^{(1)}, \quad n \in \mathbb{Z}, \tag{A.5}$$

$$(e_n)^p f_n^{(1)} (e_n)^{-p} = f_n^{(1)} - p f_{-n-1}^{(2)}, \quad (e_n)^p \bar{f}_{-n-1}^{(2)} (e_n)^{-p} = \bar{f}_{-n-1}^{(2)} + p \bar{f}_n^{(1)} \quad n \in \mathbb{Z} \tag{A.6}$$

B Appendix B

For $\hat{\omega} := \sum_{n,m} \omega_{n,m} f_n \bar{f}_m$ we have

$$\mathrm{ad}_{\hat{\omega}}\ f_i = \sum_n f_n \omega_{n,i}\ , \qquad \mathrm{ad}_{\hat{\omega}}\ \bar{f}_i = -\sum_n \omega_{i,n} \bar{f}_n$$

Then it follows

$$Ad_{\hat{\omega}}\ f_i = \sum_n f_n \left(e^{\omega}\right)_{n,i}\ , \qquad Ad_{\hat{\omega}}\ \bar{f}_i = \sum_n \left(e^{-\omega}\right)_{i,n} \bar{f}_n$$

This yields (4.15) and (4.16) which are equivalent respectively to (4.10) and (4.11).

In the same way we prove (4.18) and (4.17) which are equivalent respectively to (4.13) and (4.12).

C Appendix C

Now, let us show that

$$\tilde{S}_n(\eta) = \int\int \frac{S_n(y)}{\eta - x} d\mu(x,y), \quad \tilde{P}_n(\mu) = \int\int \frac{P_n(x)}{\mu - y} d\mu(x,y), \quad n \geq 0 \tag{C.1}$$

Proof I.

$$[\sum_{n=0}^{+\infty} d_n^{(1)} x^n, \sum_{n=0}^{+\infty} \bar{d}_n^{(1)} \eta^{-n-1}]_+ = \frac{1}{\eta}\frac{1}{1-\frac{x}{\eta}} = \sum_{n=0}^{+\infty} P_n(x)\tilde{S}_n(\eta) \tag{C.2}$$

where the first equality follows from the definitions (4.15) and (4.16), while the equality of the anti-commutator to the last member follows from (4.10)-(4.11). Multiplying both sides of the second equality by $S_n(y)d\mu(x,y)$ and integrating we come to the first equality (C.1).

We obtain the second equality (C.1) by the similar integration of

$$[\sum_{n=0}^{+\infty} d_{-n-1}^{(2)} \mu^{-n-1}, \sum_{n=0}^{+\infty} \bar{d}_{-n-1}^{(2)} y^n]_+ = \frac{1}{\mu}\frac{1}{1-\frac{y}{\mu}} = \sum_{n=0}^{+\infty} S_n(y)\tilde{P}_n(\mu) \tag{C.3}$$

(These equalities results from definitions (4.17) and (4.18), and from (4.12),(4.13)).

We shall also use (in (5.20) below) the corollary of these equalities:

$$\int\int \frac{d\mu(x,y)}{(\eta - x)(\mu - y)} = \sum_{n=0}^{\infty} \tilde{S}_n(\eta)\tilde{P}_n(\mu) =: H(\mu,\eta) \tag{C.4}$$

which is obtained by multiplying of the right hand sides of the relations (C.2) and (C.3), integrating and using the orthogonality of P_n and S_n.

Proof II. Considering the sum of entries weighted with powers of x^{-m-1} (where m is the row-number of the entry) of the n-th column of relation $K^{-1}H = B\bar{K}^{-1}$ (which follows from the known factorization relation $H\bar{K} = KB$), we obtain the first equality of (C.1) from the first of (4.5) and from the definition of bi-moments. (The second relation in (C.1) is proved similarly).

D Appendix: Links with soliton theory

In this appendix, we discuss the links between the evaluation of such symmetric rational integrals and soliton theory. There are two problems:

(A) To find which measure $d\mu$ is related to a given matrix integral? We do not know the general answer to this problem. A partial answer can be found in the papers [24], [25] and the Appendices to [26] and to [19].

(B) To find which measure $d\mu$ is related to soliton theory. This problem is addressed in [19] and [28].

We restrict ourselves to the relation to the usual (one-component) TL hierarchy.

Indeed, one can easily show that the expression

$$\tau_N(\mathbf{t}^{(1)}, \mathbf{t}^{(2)}) = \langle N, -N | g(\mathbf{t}^{(1)}, \mathbf{t}^{(2)}) | 0, 0 \rangle, \tag{D-1}$$

where $g(\mathbf{t}^{(1)}, \mathbf{t}^{(2)}) = e^{A(\mathbf{t}^{(1)}, \mathbf{t}^{(2)})}$, and A is of form

$$A(\mathbf{t}^{(1)}, \mathbf{t}^{(2)}) = \int\int \sum_{i,j=1,2} e^{V(x,\mathbf{t}^{(1)}) - V(y,\mathbf{t}^{(2)})} f^{(1)}(x) \bar{f}^{(2)}(y) d\mu_0(x,y) \tag{D-2}$$

where

$$V(x, \mathbf{t}^{(1)}) = \sum_{n=0}^{\infty} t_n^{(1)} x^n, \quad V(y, \mathbf{t}^{(2)}) = \sum_{n=0}^{\infty} t_n^{(2)} y^n \tag{D-3}$$

and where $d\mu_0(x,y)$ is a rather arbitrary measure, fits into a general expression for tau functions of the two-component KP hierarchy and also the TL hierarchy developed in [14] (see also [22], [21]).

Remark: the notations $t_m^{(1)}$ and $t_m^{(2)}$ are related respectively to the notations $-\frac{u_m}{m}$ and $\frac{v_m}{m}$ of [19]. Our $V(x, \mathbf{t}^{(j)})$ here is $V_j(x)$ of [19].

In [19] it is shown that the expression (D-1) on the one hand gives rise to multiple integrals, which, in turn, for certain choices of the measure $d\mu_0(x,y)$ (see Problem (A) above) can be identified with a partition function of some model of random matrices. On the other hand it is also an example of TL tau function.

This means that if we choose

$$d\mu(x,y) = d\mu(x,y,\mathbf{t}^{(1)}, \mathbf{t}^{(2)}) = e^{V(x,\mathbf{t}^{(1)}) - V(y,\mathbf{t}^{(2)})} d\mu_0(x,y) \tag{D-4}$$

then the expectation value (D-1) is a TL (and also two-component KP) tau function.

Remark: if $d\mu_0$ solves this or that differential equation, than, one can write a correspondent "string equation".

Notice that the first non-trivial member of the TL hierarchy was introduced and integrated in [20] and called the relativistic two-dimensional Toda lattice. In this paper a factorization problem similar to (4.3) (but with matrices that are infinite in both directions) was considered (see also [22]).

Notational Remark. In formulas below we shall denote the pair of sets of times $(\mathbf{t}^{(1)}, \mathbf{t}^{(2)})$ by a single letter $\mathbf{t}$. The only quantity which depends on a single such set (either $\mathbf{t}^{(1)}$ or $\mathbf{t}^{(2)}$) is V, where we shall point out which set it depends on.

Then from general consideration in [14] (see also [22], [21]) one finds that the integrals of rational functions $I_N(\eta, \xi, \mu, \zeta)$, see (3.9), may be obtained via the bosonization procedure as follows

$$\mathbf{I}_N(\xi, \zeta, \eta, \mu; \mathbf{t}) =$$

$$\tau_N(\mathbf{t})^{-1} \prod_{i=1}^{L_1} \xi_i^N D_1(\xi_i) \prod_{i=1}^{M_1} \eta_i^{-N} D_1(\eta_i)^{-1} \prod_{i=1}^{M_2} \mu_i^{-N} D_2(\mu_i) \prod_{i=1}^{L_2} \zeta_i^N D_2(\zeta_i)^{-1} \tau_N(\mathbf{t}), \tag{D-5}$$

where

$$D_j(z) = \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n z^n} \frac{\partial}{\partial t_n^{(j)}} \right) \tag{D-6}$$

is a vertex operator.

Notational Remark. In [1] we use the notations

$$P_n(x) = \frac{1}{\sqrt{h_n}} p_n(x), \tag{D-7}$$

$$\tilde{S}_n(x) = \sqrt{h_n} p_n^*(x) \tag{D-8}$$

$$\tilde{P}(y) = \sqrt{h_n} s_n^*(y, \mathbf{t}), \tag{D-9}$$

$$S_n(y) = \frac{1}{\sqrt{h_n}} s_n(y) \tag{D-10}$$

The functions

$$\psi_n^{(1)}(x, \mathbf{t}) := e^{V(x, \mathbf{t}^{(1)})} P_n(x, \mathbf{t}) \sqrt{h_n} = e^{V(x, \mathbf{t}^{(1)})} p_n(x, \mathbf{t}), \tag{D-11}$$

$$\psi_n^{*(1)}(x, \mathbf{t}) := e^{-V(x, \mathbf{t}^{(1)})} \frac{\tilde{S}_{n-1}(x, \mathbf{t})}{\sqrt{h_{n-1}}} = e^{-V(x, \mathbf{t}^{(1)})} p_{n-1}^*(x, \mathbf{t}) \tag{D-12}$$

are to be interpreted as first component Baker functions (respectively, adjoint first component Baker functions) which depend on a spectral parameter x, while

$$\psi_n^{(2)}(y, \mathbf{t}) := e^{V(y, \mathbf{t}^{(2)})} \frac{\tilde{P}_{n-1}(y, \mathbf{t})}{\sqrt{h_{n-1}}} = e^{V(y, \mathbf{t}^{(2)})} s_{n-1}^*(y, \mathbf{t}), \tag{D-13}$$

$$\psi_n^{*(2)}(y, \mathbf{t}) := e^{-V(y, \mathbf{t}^{(2)})} S_n(y, \mathbf{t}) \sqrt{h_n} = e^{-V(y, \mathbf{t}^{(2)})} s_n(y, \mathbf{t}) \tag{D-14}$$

are to be interpreted as second component Baker functions (respectively, adjoint second component Baker functions) which depend on a spectral parameter y.

This fact is due to the formulae (notice the factor $(-1)^N$ which result from re-ordering the fermions of two different types to achieve true sign according to (2.30))

$$\psi_n^{(1)}(x,\mathbf{t}) = (-1)^N e^{V(x,\mathbf{t}^{(1)})} \frac{\langle N+1,-N|f^{(1)}(x)g(\mathbf{t})|0,0\rangle}{\langle N,-N|g(\mathbf{t})|0,0\rangle} = x^N e^{V(x,\mathbf{t}^{(1)})} \frac{D_1(x)\tau_N(\mathbf{t})}{\tau_N(\mathbf{t})} \tag{D-15}$$

$$\psi_n^{*(1)}(x,\mathbf{t}) = (-1)^N e^{-V(x,\mathbf{t}^{(1)})} \frac{\langle N-1,-N|\bar f^{(1)}(x)g(\mathbf{t})|0,0\rangle}{\langle N,-N|g(\mathbf{t})|0,0\rangle} = x^{-N} e^{-V(x,\mathbf{t}^{(1)})} \frac{D_1(x)^{-1}\tau_N(\mathbf{t})}{\tau_N(\mathbf{t})} \tag{D-16}$$

$$\psi_n^{(2)}(y,\mathbf{t}) = e^{V(y,\mathbf{t}^{(2)})} \frac{\langle N,1-N|f^{(2)}(y)g(\mathbf{t})|0,0\rangle}{\langle N,-N|g(\mathbf{t})|0,0\rangle} = y^{-N} e^{V(y,\mathbf{t}^{(2)})} \frac{D_2(y)\tau_N(\mathbf{t})}{\tau_N(\mathbf{t})} \tag{D-17}$$

$$\psi_n^{*(2)}(y,\mathbf{t}) = e^{-V(y,\mathbf{t}^{(2)})} \frac{\langle N,-1-N|\bar f^{(2)}(y)g(\mathbf{t})|0,0\rangle}{\langle N,-N|g(\mathbf{t})|0,0\rangle} = y^{N} e^{-V(y,\mathbf{t}^{(2)})} \frac{D_2(y)^{-1}\tau_N(\mathbf{t})}{\tau_N(\mathbf{t})} \tag{D-18}$$

The first equality in each of the relations (D-15),(D-16),(D-17) and (D-18) is an example of the evaluation of I_N. Formulae (D-16) and (D-18) fit into the case $N_2 \geq N_1 \geq 0$ and are just particular cases of formula (5.9, see the Examples following (5.9)). The second equalities in each of the relations (D-15)-(D-18) are examples of bosonization formula (D-5).

Notice that different examples of I_N,

$$K_{11}(\xi,\eta,\mathbf{t}) = \frac{\langle N,-N|f^{(1)}(\xi)\bar f^{(1)}(\eta)g(\mathbf{t})|0,0\rangle}{\langle N,-N|g(\mathbf{t})|0,0\rangle} = \frac{\xi^N\eta^{-N}D_1(\xi)D_1^{-1}(\eta)\tau_N(\mathbf{t})}{\tau_N(\mathbf{t})}, \tag{D-19}$$

$$K_{22}(\mu,\zeta,\mathbf{t}) = \frac{\langle N,-N|f^{(2)}(\mu)\bar f^{(2)}(\zeta)g(\mathbf{t})|0,0\rangle}{\langle N,-N|g(\mathbf{t})|0,0\rangle} = \frac{\mu^{-N}\zeta^{N}D_2(\mu)D_2^{-1}(\zeta)\tau_N(\mathbf{t})}{\tau_N(\mathbf{t})}, \tag{D-20}$$

$$K_{12}(\xi,\zeta,\mathbf{t}) = \frac{\langle N+1,-N-1|f^{(1)}(\xi)\bar f^{(2)}(\zeta)g(\mathbf{t})|0,0\rangle}{\langle N,-N|g(\mathbf{t})|0,0\rangle} = \frac{\xi^N\zeta^{N}D_1(\xi)D_2^{-1}(\zeta)\tau_N(\mathbf{t})}{\tau_N(\mathbf{t})}, \tag{D-21}$$

$$K_{21}(\mu,\eta,\mathbf{t}) = \frac{\langle N-1,-N+1|f^{(2)}(\mu)\bar f^{(1)}(\eta)g(\mathbf{t})|0,0\rangle}{\langle N,-N|g(\mathbf{t})|0,0\rangle} = \frac{\mu^{-N}\eta^{-N}D_2(\mu)D_1^{-1}(\eta)\tau_N(\mathbf{t})}{\tau_N(\mathbf{t})}, \tag{D-22}$$

(where for the sake of brevity we omit $\mathbf{t}^{(1)},\mathbf{t}^{(2)}$-dependence in the l.h. sides), may be considered as 2-component analogue of a modified Cauchy-Baker-Akhiezer kernel, introduced in [30], [31] to present an explicit version of Segal-Wilson construction to study Virasoro

deformations of tau functions for the quasi-periodical solutions of the KP (and actually for the TL) hierarchies.

As we have obtained (see Examples following (5.9)):

$$\overset{J}{K}_{11}(\mu,\zeta) = \sum_{n=0}^{J-1} \tilde{P}_n(\mu) S_n(\zeta) + \frac{1}{\zeta - \mu} \tag{D-23}$$

$$\overset{J}{K}_{22}(\xi,\eta) = \sum_{n=0}^{J-1} P_n(\xi) \tilde{S}_n(\eta) + \frac{1}{\xi - \eta} \tag{D-24}$$

$$\overset{J}{K}_{21}(\xi,\zeta) = \sum_{n=0}^{J-1} P_n(\xi) S_n(\zeta) \tag{D-25}$$

$$\overset{J}{K}_{12}(\mu,\eta) = \sum_{n=0}^{J-1} \tilde{P}_n(\mu) \tilde{S}_n(\eta) - \int \frac{d\mu(x,y)}{(\eta - x)(\mu - y)} \tag{D-26}$$

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